MAP555 - Signal processing- Exercise set 2
Digital Signal Processing

Exercise 1 Sampling

1. What is the minimal sampling frequency for using the Nyquist/Shannon theorem with signal

$$x(t) = \frac{\sin(t)}{t} = \mathrm{sinc}(t)$$

2. Let $x(t)$ be a signal of frequency support $[-B, B]$. What is the maximum frequency contained in the following signals:
   i. $\frac{dx(t)}{dt}$
   ii. $x(t)cos(2\pi f_0 t)$
   iii. $x(2t)$
   iv. $x^2(t)$
3. Let $x(t)$ be a signal of frequency support $[-B, B]$.
   i. $x(t)$ is regularly sampled over a measurement time of $[0, \frac{N-1}{N}L]$ resulting in N samples. What is the relation between N, L and B ensuring that the signal can be reconstructed in the interval?
   ii. When the finite sampled signal is considered periodic, its spectrum has a mass only on a finite locations, express these locations in frequencies.

Exercise 2 Fourier Transform

1. Compute the DTFT of the following discrete signals:
   i. $x[n] = \delta[n]$
   ii. $x[n] = \delta[n] + 6\delta[n-1] + 3\delta[n-2]$
   iii. $x[n] = \delta[n] + \delta[n-1] + \delta[n+1]$
   iv. $x[n] = a^n \Gamma[n]$ with $|a| < 1$
2. Let x be a finite signal of N samples with $x[n] = cos(2\pi \frac{k_0}{N} n)$ with $k_0 \in \mathbb{Z}$ a be finite signal of N samples.
   i. Express the DFT of $x[n]$ using the Euler formula.
   ii. Use the following geometric formula to compute $X[k]$ by separating the cases $k = \pm k_0$ and $k \neq \pm k_0$

$$\sum_{n=0}^{N-1} a^n = \frac{1 - a^N}{1 - a}$$

   iii. What would happen if k_0 was not an integer (no need to compute)?

Exercise 3 Signal linear interpolation

We have access to de discrete time signal $x[n]$ but would like interpolate its values. To this end we define the signal $y[n]$ such that

$$y[n] = \begin{cases} x[n/2] & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

If we replace the null samples by the average of the values surrounding it we get the following interpolated signal:

$$z[n] = \begin{cases} y[n] & \text{when } n \text{ is even} \\ \frac{1}{2}(y[n-1] + y[n+1]) & \text{when } n \text{ is odd} \end{cases}$$

1. Express the Discrete time Fourier transform of $y[n]$ as a function of the DTFT of $x[n]$. Plot a schematic representation of $|X(e^{2i\pi f})|$ and $|Y(e^{2i\pi f})|$.
2. Show that the relation between $y[n]$ and $z[n]$ is linear and recover the corresponding impulse response $h[n]$.
3. Compute the DTFT of $h[n]$ and plot its magnitude.
4. What is the role of the interpolator filter in frequency domain ?

Exercise 4 Finite Impulse Response (FIR) filter

Let H be a filter of impulse response

$$h[n] = \begin{cases} a & \text{if } n \in \{0, 3\} \\ b & \text{if } n \in \{1, 2\} \\ 0 & else \end{cases}$$

1. Compute the DTFT of the filter and prove that the filter is linear in frequency : it can be expressed as $H(e^{2i\pi f}) = e^{2i\pi f T_0} A(f)$.
2. For which values of a, b the filter is a high-pass filter?