MAP555 - Signal processing- Exercise set 3
Random Signals

Exercise 1

Let X be a stochastic process $X(t) = A\cos(\omega t + \varphi)$ where φ is a uniform random variable on $[0, 2\pi]$.

1. Compute $\mathsf{E}[X(t)]$ and $\mathsf{E}[X(t_1)X(t_2)]$. Is the signal $X(t)$ wide sens stationary?
2. Express the autocorrelation function $R_{xx}(\tau)$.
3. What is the average power of $X(t)$?

Exercise 2

We define the signal $X[n]$ as :

$$X[n] = Z[n] + \theta z[n-1],\ n \in \mathbb{Z}$$

where $z(n)$ is and IID noise such that $z(n) \sim \mathcal{N}(0, \sigma^2)$.

1. What is the relation between $X[n]$ and $Z[n]$? Express the impulse response of the corresponding system.
2. Compute $m_X[n] = E[X(n)]$.
3. Compute the autocorrelation of $X[n]$. Is the signal wide sens stationary?

Exercise 3

We define the stochastic process $X[n]$ as :

$$X[n] = \phi X[n-1] + \theta Z[n],\ n \in \mathbb{Z}$$

where $Z[n]$ is IID with $Z[n] \sim \mathcal{N}(0, \sigma^2)$ and $Z[s]$ is independent from $X[n]$ for $s > n$ (causality).

1. Express $X[n]$ as a function of $X|0]$ and $Z[k]$ for $k \leq n$.
2. Compute the means of $X[n]$, $n > 0$ as a function of $E[X[0]]$. Show that if $|\phi| < 1$ this means converges to the constant 0.
3. We suppose now that $X[n]$ is WSS. We will recover the autocorrelation form the recurrence relation of $X[n]]$ given above. To do that multiply the equation above by $X[n+q]$ for $q > 0$ and compute its expectation.
4. Do the same thing with $q = 0$. Deduce the value of teh Autocorrelation $R_x[q]$ $\forall q$.
5. Compute the PSD $S_x(e^{2i\pi f})$ of X from its autocorrelation.
6. Compute the PSD $S_x(e^{2i\pi f})$ of X directly using the convolution formula. Which was the easiest to compute?

Exercise 4

Let X be a stochastic process derived from a Poisson process $X_p(t)$ as

$$X(t) = \begin{cases} A & \text{if } X_p \text{ is odd} \\ -A & \text{if } X_p \text{ is even} \end{cases} \tag{1}$$

where $A \sim \mathcal{U}(\{1, -1\})$ is a uniform random variable. This random process is WSS and its autocorrelation can be expressed as :

$$R_X(\tau) = A^2 e^{-2\lambda|\tau|} \tag{2}$$

1. Plot 3 realizations of X. plot the autocorrelation function.
2. Compute the PSD of the signal and plot it.
3. Let $Y(t)$ be a stochastic process depending on some Gaussian IID noise $B(t)$ of variance σ^2.

$$Y(t) = \tilde{X}(t) + B(t) \tag{3}$$

   Compte the SNR for $Y(t)$.
4. We apply to $Y(t)$ a filter of Transfer function:

$$H(f) = \begin{cases} 1 & \text{si } |f| < f_c \\ 0 & \text{sinon} \end{cases} \tag{4}$$

   Compute the average power of the noise after filtering.
5. Compute the average power of the signal $X(t)$ after filtering.
6. Compute the SNR of $Y(t)$ after filtering supposing that $X(t)$ and $B(t)$ are both centered and independent. plot the SNR as a function of f_c.
7. What value of f_c maximizes the SNR while preserving at least 1/2 of the signal average power.