(Quasi-)Newton methods

Alexandre Gramfort

Master 2 Data Science, Univ. Paris Saclay
Optimisation for Data Science

Table of Contents

Outline

So far you have seen:

- gradient descent
- proximal gradient descent
- accelerated gradient descent
- (proximal) coordinate descent
- conjugate gradient

Now

- Newton methods
- Quasi-Newton methods
- Methods dedicated to non-linear least squares

Quasi-Newton and in particular L-BFGS are still heavily used to tackle smooth potentially large scale optim problems in machine learning (e.g. ℓ_2 logistic regression, conditional random fields)

Not seen in the course:

- Prox-Newton methods for the twice differentiable + proximable penalty case
- Constrained methods (x is constrained to a subset of $\mathbb{R}^n$)
- Stochastic quasi-Newton methods (when f is a sum)

Remark: State-of-the-art solvers like `liblinear` are combining Prox-Newton and coordinate descent methods for logistic regression.

This course is largely based on the book:

- Wright and Nocedal, Numerical Optimization, 1999, Springer, Chapters 6 and 8.

Newton method

It is used to find the zeros of a differentiable non-linear function g:

$$\text{Find } x \text{ such that } g(x) = 0, \text{ where } g : \mathbb{R}^n \to \mathbb{R}^n.$$

Given a starting point x_0, Newton method consists in iterating:

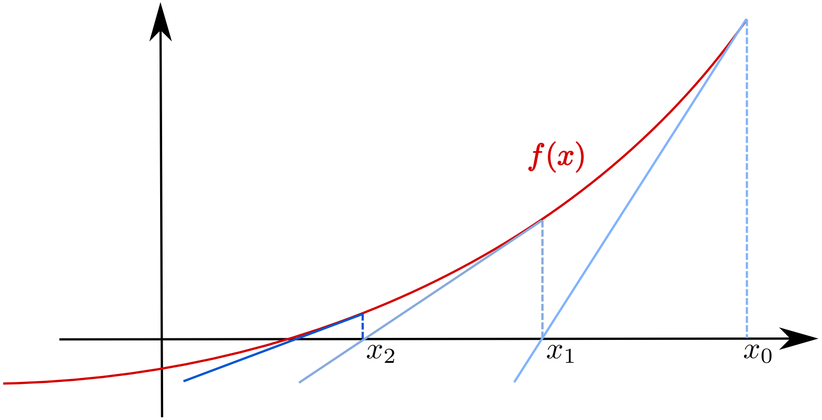
$$x_{k+1} = x_k - g'(x_k)^{-1} g(x_k)$$

where $g'(x)$ is the derivative (Jacobian) of g at point x.

We have that:

- $g'(x_k)$ is matrix in $\mathbb{R}^{n \times n}$
- each iteration requires to solve a linear system.

Newton method in 1d



Newton method?

Applying this method to the optimization problem:

$$\min_{x \in \mathbb{R}^n} f(x)$$

consists in setting $g(x) = \nabla f(x)$, i.e., looking for stationary points (i.e. $\nabla f(x) = 0$).

The iterations read:

$$x_{k+1} = x_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k) \ .$$

Newton method is particularly interesting as its convergence is quadratic locally around x^*, i.e.:

$$\|x_{k+1} - x^*\| \leq \gamma \|x_k - x^*\|^2, \gamma > 0 \ .$$

Finding Newton's algorithm

Assuming f is twice differentiable, the Taylor expansion at order 2 of f at x reads:

$$\forall h \in \mathbb{R}^n,\ f(x+h) = \underbrace{f(x) + \nabla f(x)^\top h + \frac{1}{2} h^\top \nabla^2 f(x) h}_{Q_x(h)} + o(\|h\|^2)$$

Exercise

Can you minimize $Q_x(h)$ with respect to h?

Convergence of Newton method

Theorem (Convergence of Newton method)

Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ assumed twice differentiable $\mathcal{C}^2$, and $x^* \in \mathbb{R}^n$ an isolated zero of g ($g(x^*) = 0$). Assuming that $g'(x^*)$ is invertible, there exists a closed ball $\mathcal{B}$ centered on x^*, such that for every $x_0 \in \mathcal{B}$, the sequence x_k obtained with Newton algorithm stays in $\mathcal{B}$ and converges towards x^*. Furthermore, there is a constant $\gamma > 0$, such that $\|x_{k+1} - x^*\| \leq \gamma \|x_k - x^*\|^2$.

$\rightarrow$ See proof in lecture notes.

Remark: Convergence of Newton is local. The method may diverge if the initial point is too far from x^*

Remark: That is why Newton should be coupled with a line search strategy:

$$x_{k+1} = x_k - \rho_k \nabla^2 f(x_k)^{-1} \nabla f(x_k)$$

where $\rho_k > 0$ is a stepsize found by line search (Wolfe conditions).

Newton on quadratic function

Exercise

Show that for a quadratic function

$$f(x) = \frac{1}{2}x^\top Ax - b^\top x + c,\ x \in \mathbb{R}^n$$

with A symmetric positive definite, Newton method converges in one iteration independently of the choice of x_0.

Remark: Newton is therefore not affected by the conditioning of the problem (not like Gradient descent).

$\rightarrow$ See notebook.

Newton on a non-convex problem

- Newton's method finds the stationnary points ($\nabla f = 0$).
- It is attracted to saddle points.
- Newton's direction may not be a descent direction:

$$\nabla f(x_k)^\top \left[(\nabla^2 f(x_k))^{-1} \nabla f(x_k)\right] < 0$$

- To guarantee one has a descent direction one needs to regularize the Hessian and in practice one needs to use a line search:

$$x_{k+1} = x_k - \rho_k (\nabla^2 f(x_k) + \lambda \mathrm{I}_n)^{-1} \nabla f(x_k)$$

  where $\lambda > 0$ is the regularization parameter and ρ_k is a stepsize found by line search.

Remark: line search is mandatory also for convex problems.

Table of Contents

Variable metric

The idea behind variable metric methods consists in using iterations of the form

$$\begin{cases} d_k = -B_k g_k \ , \\ x_{k+1} = x_k + \rho_k d_k \ , \end{cases}$$

where $g_k = \nabla f(x_k)$, B_k is a positive definite matrix and $\rho_k \geq 0$ is a step size.

Remark: If B_k is a positive definite matrix $-B_k g_k$ is a descent direction.

$\rightarrow$ If $B_k = I_n$, it corresponds to gradient descent.
$\rightarrow$ Setting $B_k = B$ is the fixed metric case.

Fixed metric case

When minimizing

$$\min_{x \in \mathbb{R}^n} f(x)$$

one can set $x = Cy$ with C invertible (change of variable). Let us denote $\tilde{f}(y) = f(Cy)$. This leads to:

$$\nabla \tilde{f}(y) = C^\top \nabla f(Cy) \enspace .$$

Gradient descent applied to $\tilde{f}(y)$ reads:

$$y_{k+1} = y_k - \rho_k C^\top \nabla f(Cy_k)$$

which means using $B = CC^\top$ as it is equivalent to:

$$x_{k+1} = x_k - \rho_k CC^\top \nabla f(x_k) \enspace .$$

Question: How would you choose C for quadratic problem?

Quadratic case

Theorem (Preconditioned gradient descent)

Let $f(x)$ a positive definite quadratic form with Hessian A, and B a positive definite matrix. The preconditioned gradient algorithm:

$$\begin{cases} x_0 = \textit{fixed}, \\ x_{k+1} = x_k - \rho_k B g_k, \ \ \rho_k \ \textit{optimal} \end{cases}$$

has a linear convergence: $\|x_{k+1} - x^*\| \leq \gamma \|x_k - x^*\|$
where:

$$\gamma = \frac{\chi(BA) - 1}{\chi(BA) + 1} < 1 \ .$$

$\chi(M) = \lambda_1 / \lambda_n$ is the Euclidian conditioning *i.e.*, ratio of largest and lowest eigenvalues (≥ 1).

Quadratic case

So we have a linear convergence:

$$\|x_{k+1} - x^*\| \leq \gamma \|x_k - x^*\|$$

where:

$$\gamma = \frac{\chi(BA) - 1}{\chi(BA) + 1} < 1 \ .$$

Remark

The lower the conditioning of BA, the faster is the algorithm. One cannot set $B = A^{-1}$ as it would imply having already solved the problem, but this however suggests to use B so that it approximates A^{-1}. This is the idea behind quasi-Newton methods.

Drawbacks of Newton's method

- Quadratic convergence is an interesting property, but most of the time, Newton's method is too costly!
- Computing the Hessian is n times more costly in time and memory than the gradient
- If the problem is non-convex, regularization is hard and costly
- Then, one needs to compute $H^{-1}\nabla f(x) \rightarrow O(n^3)$
- What if $n = 10^3$?

Idea of quasi-Newton methods:

mimic Newton's direction without the computational load.

Table of Contents

Quasi-Newton

A quasi-Newton method reads

$$\begin{cases} d_k = -B_k g_k \ , \\ x_{k+1} = x_k + \rho_k d_k \ , \end{cases}$$

or

$$\begin{cases} d_k = -H_k^{-1} g_k \ , \\ x_{k+1} = x_k + \rho_k d_k \ , \end{cases}$$

where B_k (resp. H_k) is a matrix which aims to approximate the inverse of the Hessian (resp. the Hessian) of f at x_k.

Question: How to achieve this?

Quasi-Newton

One can start with $B_0 = I_n$. how to update B_k at every iteration?

Idea: apply a Taylor expansion on the gradient, notice that at point x_k, the gradient and the Hessian are such that:

$$g_{k+1} = g_k + \nabla^2 f(x_k)(x_{k+1} - x_k) + \epsilon(x_{k+1} - x_k) \ .$$

Towards convergence one should have:

$$\boxed{g_{k+1} - g_k \approx \nabla^2 f(x_k)(x_{k+1} - x_k) \ .}$$

Quasi-Newton relation (or secant condition)

Definition (Quasi-Newton relation)

Two matrices B_{k+1} and H_{k+1} verify the quasi-Newton relation (or secant condition) if:

$$H_{k+1}(x_{k+1} - x_k) = \nabla f(x_{k+1}) - \nabla f(x_k)$$

or

$$x_{k+1} - x_k = B_{k+1}(\nabla f(x_{k+1}) - \nabla f(x_k))$$

Problem: How to update B_k keeping it positive definite?

Update formula of Hessian

The update strategy at iteration

$$\begin{cases} d_k = -B_k g_k \ , \\ x_{k+1} = x_k + \rho_k d_k \ , \end{cases}$$

is to correct B_k with a symmetric matrix Δ_k:

$$B_{k+1} = B_k + \Delta_k$$

such that the quasi-Newton relation (secant condition) holds:

$$x_{k+1} - x_k = B_{k+1}(g_{k+1} - g_k)$$

with B_{k+1} positive definite, assuming B_k is positive definite.

Idea: Use rank 1 or 2 matrices for Δ_k

Broyden formula (known as SR1)

Let's consider a rank 1 correction on the Hessian:

$$H_{k+1} = H_k + \sigma v v^\top \quad , \quad \sigma = \pm 1, v \in \mathbb{R}^n$$

The matrix H_{k+1} should verify the secant condition: $y_k = H_{k+1} s_k$, where $y_k = g_{k+1} - g_k$ and $s_k = x_{k+1} - x_k$. It follows that:

$$y_k = H_k s_k + (\sigma v^\top s_k) v \Rightarrow \exists \delta \in \mathbb{R}, v = \delta(y_k - H_k s_k)$$

Using the equality it leads to:

$$y_k - H_k s_k = \sigma \delta^2 [s_k^\top (y_k - H_k s_k)](y_k - H_k s_k)$$

this imposes that:

$$\sigma = \mathrm{sign}[s_k^\top (y_k - H_k s_k)] \quad \delta = \pm |s_k^\top (y_k - H_k s_k)|^{-1/2}$$

This leads to:

$$H_{k+1} = H_k + \frac{(y_k - H_k s_k)(y_k - H_k s_k)^\top}{(y_k - H_k s_k)^\top s_k}$$

Broyden formula (known as SR1)

Starting from:

$$H_{k+1} = H_k + \frac{(y_k - H_k s_k)(y_k - H_k s_k)^\top}{(y_k - H_k s_k)^\top s_k}$$

and using the matrix inversion lemma (Woordbury-Sherman-Morrison) leads to:

$$\boxed{B_{k+1} = B_k + \frac{(s_k - B_k y_k)(s_k - B_k y_k)^\top}{(s_k - B_k y_k)^\top y_k} \quad ,}$$

also known as Broyden or SR1 formula.

Broyden formula

Theorem

Let f a quadratic form positive definite. Let us consider the method that, starting for x_0, iterates:

$$x_{k+1} = x_k + s_k \quad ,$$

where the vectors s_k are linearly independent. Then the sequence of matrices starting by B_0 and defined as:

$$B_{k+1} = B_k + \frac{(s_k - B_k y_k)(s_k - B_k y_k)^\top}{(s_k - B_k y_k)^\top y_k} \quad ,$$

where $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$, converges in less than n iterations towards A^{-1}, the inverse of the Hessian of f.

$\rightarrow$ Cf. proof in lecture notes

Remark: No guarantee that the matrices B_k are positive definite, even if the function f is quadratic and $B_0 = I_n$ ($\sigma = -1$).

Davidon, Fletcher and Powell formula

Using a rank 2 correction, it reads:

$$B_{k+1} = B_k + \alpha u u^\top + \beta v v^\top \quad .$$

Imposing the quasi-Newton relation (secant condition):

$$\begin{aligned} B_{k+1} y_k &= s_k \\ \Rightarrow B_k y_k + \alpha(u^\top y_k)u + \beta(v^\top y_k)v &= s_k \\ \Rightarrow \alpha(u^\top y_k)u + \beta(v^\top y_k)v &= s_k - B_k y_k \end{aligned}$$

This equation has not a unique solution. The choice for u and v by DFP is:

$$u = s_k \quad \text{and} \quad v = B_k y_k$$

Davidon, Fletcher and Powell formula

Solving for α and β the equation:

$$\alpha(s_k^\top y_k)s_k + \beta(y_k^\top B_k y_k)B_k y_k = s_k - B_k y_k$$

we obtain

$$\alpha = \frac{1}{s_k^\top y_k} \quad \text{and} \quad \beta = -\frac{1}{y_k^\top B_k y_k}$$

Davidon, Fletcher and Powell formula

The DFP formula is a rank 2 correction. It reads:

$$\boxed{B_{k+1} = B_k + \frac{s_k s_k^\top}{s_k^\top y_k} - \frac{B_k y_k y_k^\top B_k}{y_k^\top B_k y_k} \quad . \qquad (1)}$$

Theorem

Let us consider the update

$$\begin{cases} d_k = -B_k g_k \ , \\ x_{k+1} = x_k + \rho_k B_k g_k, \ \ \rho_k \ \textit{optimal} \end{cases}$$

where B_0 is positive definite and provided as well as x_0. Then the matrices B_k defined as in (1) are positive definite for all $k > 0$.

$\rightarrow$ Cf. proof in lecture notes

Davidon-Fletcher-Powell algorithm

```
Require: $\varepsilon > 0$ (tolerance), $K$ (maximum number of iterations)
 1: $x_0 \in \mathbb{R}^n$, $B_0 > 0$ (for example $I_n$)
 2: for $k = 0$ to $K$ do
 3:     if $\|g_k\| < \varepsilon$ then
 4:         break
 5:     end if
 6:     $d_k = -B_k \nabla f(x_k)$
 7:     $x_{k+1} = x_k + \rho_k d_k$ (Compute optimal step size $\rho_k$)
 8:     $s_k = \rho_k d_k$
 9:     $y_k = g_{k+1} - g_k$
10:     $B_{k+1} = B_k + \frac{s_k s_k^\top}{s_k^\top y_k} - \frac{B_k y_k y_k^\top B_k}{y_k^\top B_k y_k}$
11: end for
12: return $x_{k+1}$
```

Remark: In Numpy to do things like $s_k s_k^\top$ use the `np.outer` function.

Davidon-Fletcher-Powell algorithm

This algorithm has a remarkable property when the function f is quadratic.

Theorem

When f is a quadratic form, the algorithm of Davidon-Fletcher-Powell generates a sequence of directions $s_0, \dots, s_k$ which verify:

$$\begin{aligned} s_i A^\top s_j &= 0, \quad 0 \le i < j \le k, \\ B_{k+1} A s_i &= s_i, \qquad\quad 0 \le i \le k. \end{aligned} \tag{2}$$

Remark: This theorem says that in the quadratic case, the algorithm is like a conjugate gradient method, which therefore converges in at most n iterations.

Remark: This required to have an optimal step size.

Davidon-Fletcher-Powell algorithm

One can also notice that for $k = n - 1$

$$B_n A s_i = s_i, i = 0, \dots, n-1,$$

and since all s_i are linearly independent it implies $B_n = A^{-1}$.

Remark: One can show that in the general case (non-quadratic), if the direction d_k is reinitialized to $-g_k$ periodically, this algorithm converges to a local minimum $\hat{x}$ of f and that:

$$\lim_{k \to \infty} B_k = \nabla^2 f(\hat{x})^{-1} \ .$$

This implies that close to the optimum, the method behaves like a Newton method. This justifies the use of $\rho_k = 1$ when using approximate line search.

Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm

- The BFGS formula is derived from the formula of DFP by swapping the roles of s_k and y_k.
- The formula obtained allows to maintain an approximation H_k of the Hessian which satisfies the same properties: $H_{k+1} > 0$ if $H_k > 0$ and satisfying the quasi-Newton relation:

$$y_k = H_{k+1} s_k \quad .$$

- The BFGS formula therefore reads:

$$\boxed{H_{k+1} = H_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{H_k s_k s_k^\top H_k}{s_k^\top H_k s_k} \quad .}$$

Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm

BFGS formula:

$$H_{k+1} = H_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{H_k s_k s_k^\top H_k}{s_k^\top H_k s_k} \quad .$$

Exercise

Use Sherman-Morrison formula: $(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}uv^\top A^{-1}}{1+v^\top A^{-1}u}$ *to derive an update of* B_{k+1}.

Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm

BFGS formula:

$$H_{k+1} = H_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{H_k s_k s_k^\top H_k}{s_k^\top H_k s_k} \quad .$$

Exercise

Use Sherman-Morrison formula: $(A + uv^\top)^{-1} = A^{-1} - \frac{A^{-1}uv^\top A^{-1}}{1+v^\top A^{-1}u}$ *to derive an update of* B_{k+1}*.*

$$B_{k+1} = (\mathrm{I}_n - \mu_k s_k y_k^\top) B_k (\mathrm{I}_n - \mu_k y_k s_k^\top) + \mu_k s_k s_k^\top , \mu_k = \frac{1}{y_k^\top s_k}$$

Remark: DFP and BFGS have the same computational cost.

Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm

```
Require: $\varepsilon > 0$ (tolerance), $K$ (maximum number of iterations)
 1: $x_0 \in \mathbb{R}^n$, $H_0 > 0$ (for example $I_n$)
 2: for $k = 0$ to $K$ do
 3:     if $\|g_k\| < \varepsilon$ then
 4:         break
 5:     end if
 6:     $d_k = -H_k^{-1} \nabla f(x_k)$
 7:     $x_{k+1} = x_k + \rho_k d_k$ (optimal step size $\rho_k$ with line search)
 8:     $s_k = \rho_k d_k$
 9:     $y_k = g_{k+1} - g_k$
10:     $H_{k+1} = H_k + \frac{y_k y_k^\top}{y_k^\top s_k} - \frac{H_k s_k s_k^\top H_k}{s_k^\top H_k s_k}$
11: end for
12: return $x_{k+1}$
```

Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm

The BFGS algorithm has the same property as the DFP method:

- in the quadratic case it produces conjugate directions
- it converges in less than n iterations and $H_n = A$
- Usually combined with Wolfe or Goldstein's rule.

but:

- much less sensitive than DFP to the use of approximate step size (to combine with Wolfe or Goldstein's rule).

Remark: BFGS is in scipy see `scipy.optimize.fmin_bfgs`.

Limited-memory BFGS (L-BFGS) algorithm

- L-BFGS is a variant of BFGS that limits memory usage. It was originally proposed by Liu and Nocedal in 1989:
- Does not store matrix of the size of the Hessian, $n \times n$ which can be prohibitive in applications such as computer vision or machine learning where n can be millions.
- L-BFGS stores only a few vectors that are used to approximate the matrix H_k^{-1}
- So the memory usage is linear in the dimension of the problem.

[Liu, D. C.; Nocedal, J. (1989). "On the Limited Memory Method for Large Scale Optimization". Mathematical Programming B. 45 (3): 503–528.]

Limited-memory BFGS (L-BFGS) algorithm

- L-BFGS is an algorithm of the quasi-Newton family with $d_k = -B_k \nabla f(x_k)$.
- Difference is in the computation of the product between B_k and $\nabla f(x_k)$.
- Idea is to keep in memory the last low rank corrections, more specifically the last m values of $s_k = x_{k+1} - x_k$ and $y_k = g_{k+1} - g_k$.
- Use m times recursively the formula:

$$B_{k+1} = (\mathrm{I}_n - \mu_k s_k y_k^\top) B_k (\mathrm{I}_n - \mu_k y_k s_k^\top) + \mu_k s_k s_k^\top, \mu_k = \frac{1}{y_k^\top s_k}$$

but never storing in memory a matrix $n \times n$.

Limited-memory BFGS (L-BFGS) algorithm

Let $\mu_k = \frac{1}{y_k^\top s_k}$, the algorithm to obtain d_k reads:

```
Require: $m$ (memory size)
 1: $q = g_k$
 2: for $i = k - 1$ to $k - m$ do
 3:     $\alpha_i = \mu_i s_i^\top q$
 4:     $q = q - \alpha_i y_i$
 5: end for
 6: $z = B_k^0 q$
 7: for $i = k - m$ to $k - 1$ do
 8:     $\beta = \mu_i y_i^\top z$
 9:     $z = z + s_i(\alpha_i - \beta)$
10: end for
11: $d_k = -z$
```

where B_k^0 is positive definite matrix, e.g., a diagonal matrix, so that initially setting z is fast.

Limited-memory BFGS (L-BFGS) algorithm

- Like BFGS, L-BFGS does not need exact line search to converge.
- L-BFGS is for smooth unconstrained problem but can be extended to handle simple box constraints (a.k.a. bound constraints): $l_i \leq x_i \leq u_i$ where l_i and u_i are per-variable constant lower and upper bounds. This algorithm called L-BFGS-B is due to Byrd et al. (1995).
- L-BFGS-B in scipy as `scipy.optimize.fmin_l_bfgs_b`.

[Byrd, R. H.; Lu, P.; Nocedal, J.; Zhu, C. (1995). "A Limited Memory Algorithm for Bound Constrained Optimization". SIAM J. Sci. Comput. 16 (5): 1190–1208. doi:10.1137/0916069.]

$\rightarrow$ Can you solve a Lasso with L-BFGS-B?

Table of Contents

non-linear least-squares

The function to minimize reads:

$$f(x) = \frac{1}{2} \sum_{i=1}^{m} f_i(x)^2 \ .$$

Newton method can be applied to the minimization of f. The gradient and the Hessian matrix read in this particular case:

$$\nabla f(x) = \sum_{i=1}^{m} f_i(x) \nabla f_i(x) \ ,$$

and

$$\nabla^2 f(x) = \sum_{i=1}^{m} \nabla f_i(x) \nabla f_i(x)^\top + \sum_{i=1}^{m} f_i(x) \nabla^2 f_i(x) \ .$$

Gauss-Newton method

Idea is to ignore the second order terms. The Hessian reads:

$$H(x) \approx \sum_{i=1}^{m} \nabla f_i(x) \nabla f_i(x)^\top \ .$$

This matrix is always positive. Furthermore when m is much larger than n, this matrix is often positive definite.

The Gauss-Newton method uses this approximation of $H(x)$ in a Newton-like solver:

$$\begin{cases} x_0 = \text{fixed}, \\ H_k = \sum_{i=1}^{m} \nabla f_i(x_k) \nabla f_i(x_k)^\top, \\ x_{k+1} = x_k - H_k^{-1} \nabla f(x_k) \ . \end{cases}$$

Gauss-Newton method

To guarantee the convergence of the Gauss-Newton method, it can be combined with a line search procedure:

$$\begin{cases} x_0 = \text{fixed}, \\ H_k = \sum_{i=1}^{m} \nabla f_i(x_k) \nabla f_i(x_k)^\top, \\ x_{k+1} = x_k - \rho_k H_k^{-1} \nabla f(x_k) \end{cases} .$$

Levenberg-Marquardt method

- Levenberg-Marquardt method is a variant of Gauss-Newton that enforces that the Hessian approximation H_k is positive definite.
- The idea is simply to replace H_k by $H_k + \lambda I_n$.

$$\begin{cases} x_0 = \text{fixed}, \\ H_k = \sum_{i=1}^{m} \nabla f_i(x_k) \nabla f_i(x_k)^\top, \\ d_k = -(H_k + \lambda I_n)^{-1} \nabla f(x_k) \\ x_{k+1} = x_k + \rho_k d_k \quad . \end{cases}$$

- If λ is large, method is equivalent to a gradient method.
- The Levenberg-Marquardt method in scipy as `scipy.optimize.leastsq`.

References

- Wright and Nocedal, Numerical Optimization, 1999, Springer, Chapters 6 and 8.